Nonlinear Analysis: Modelling and Control, Vilnius, IMI, Vol. **2**, p. 43-58 (1998)

MAPS FOR ANALYSIS OF NONLINEAR DYNAMICS

B. Kaulakys

Institute of Theoretical Physics and Astronomy, A. Goštauo 12, 2600 Vilnius, Lithuania and Vilnius University, Department of Quantum Electronics

Area preserving maps provide the simplest and most accurate means to visualize and quantify the behavior of nonlinear systems. Convenience of the mapping equations of motion for investigation of transition to chaotic behavior in dynamics of classical atom in microwave field, transition to nonchaotic behavior in randomly driven systems and induced quantum dynamics of simple and multilevel systems is demonstrated.

1 Introduction

A common method of displaying the dynamics is through a Poincaré section. The Poincaré section is a device invented by Henri Poincaré as a means of simplifying phase space diagrams of complicated systems. It is constructed by viewing the phase space diagram stroboscopically in such a way that the motion is observed periodically. A dynamical system whose phase space is tree-dimensional may be converted through the Poincaré section to a two-dimensional mapping. Sometimes such mappings may be further simplified to one-dimensional. Because of their relative simplicity one- and two-dimensional maps provide several advantages over the differential equations. They allow for simple reveal of many characteristics of chaotic behavior, such as sensitivity to initial conditions, illustration the mechanisms of bifurcation and so on.

Area preserving maps provide the simplest and most accurate means to visualize and quantify the behavior of conservative systems. Such maps may be iterated on even the simple computer. For analysis of the non-conservative systems sometimes the non-area-preserving maps may be introduced.

Here we demonstrate examples of investigation of transition to chaotic behavior in the dynamics of classical atom in microwave field by area-preserving maps, transition to non-chaotic behavior in randomly driven systems by the non-area-preserving maps and induced quantum dynamics of simple and multilevel systems represented by the appropriate quantum maps.

2 Kepler maps

Consider dynamics of the classical hydrogenic atom in the monochromatic field. The direct way of coupling the electromagnetic field to the electron Hamiltonian is through the $\mathbf{A}\cdot\mathbf{P}$ interaction, where $\mathbf{A}$ is the vector potential of the field and $\mathbf{P}$ is the generalized momentum of the electron. The Hamiltonian of the hydrogen atom in a linearly polarized field $F\cos(\omega t+\vartheta)$, with F, ω and ϑ being the field strength amplitude, field frequency and phase, respectively, in atomic units is

$$H=\frac{1}{2}\left(\mathbf{P}-\frac{\mathbf{F}}{\omega}\sin(\omega t+\vartheta)\right)^2-\frac{1}{r}. \tag{2.1}$$

Electron energy change due to interaction with the external field follows from the Hamiltonian equations of motion [1]

$$\dot{E}=-\dot{\mathbf{r}}\cdot\mathbf{F}\cos(\omega t+\vartheta). \tag{2.2}$$

Using parametric equations of motion in the Coulomb field we may calculate the change of the electron's energy in the classical perturbation theory approximation.

Measuring the time of the field action in the field periods one may introduce the scale transformation where the scaled field strength and the scaled energy are $F_s=F/\omega^{4/3}$ and $E_s=E/\omega^{2/3}$, respectively. However, it is convenient [2, 3] to introduce the positive scaled energy $\varepsilon=-2E_s$ and the relative field strength $F_0=Fn_0^4=F_s/\varepsilon_0^2$, with n_0 being the initial effective principle quantum number, $n_0=(-2E_0)^{-1/2}$. The threshold values of the relative field strength F_0 for the ionization onset depends weaker upon the initial effective principle quantum number n_0 and the relative frequency of the field $s_0=\omega n_0^3$ than the scaled field strength F_s.

We restrict our subsequent consideration to the one-dimensional model, which corresponds to the states of low orbital quantum numbers $l\ll n$ and is widely used in theoretical analysis [4–8]. Integration of Eq. (2.2) for motion between two subsequent passages at the aphelion (where $\dot{x}=0$ and there is no energy exchange between the field and the atom) results to the map (see [9–11] for details)

$$\begin{cases} \varepsilon_{j+1}= & \varepsilon_j-\pi F_0\varepsilon_0^2 h\left(\varepsilon_{j+1}\right)\sin\vartheta_j, \\ \vartheta_{j+1}= & \vartheta_j+2\pi\varepsilon_{j+1}^{-3/2}-\pi F_0\varepsilon_0^2\eta\left(\varepsilon_{j+1}\right)\cos\vartheta_j \end{cases} \tag{2.3}$$

where

$$h\left(\varepsilon_{j+1}\right)=\frac{4}{\varepsilon_{j+1}}\mathbf{J}'_{s_{j+1}}(s_{j+1}). \tag{2.4}$$

Here $s\equiv\varepsilon^{-3/2}=\omega/(-2E)^{3/2}=\omega/\Omega$ is the relative frequency of the field, i.e., the ration of the field frequency ω to the Kepler orbital frequency $\Omega=(-2E)^{3/2}$, and $\mathbf{J}'_s(z)$ is the derivative of the Anger function with respect to the argument z. The function $\eta\left(\varepsilon_{j+1}\right)$ may be obtained from requirement of area-preserving of the map (2.3)

$$\frac{\partial\left(\varepsilon_{j+1},\vartheta_{j+1}\right)}{\partial\left(\varepsilon_j,\vartheta_j\right)}=1. \tag{2.5}$$

This requirement yields

$$\eta\left(\varepsilon_{j+1}\right)=\frac{dh\left(\varepsilon_{j+1}\right)}{d\varepsilon_{j+1}}. \tag{2.6}$$

The derivative of the Anger function

$$\mathbf{J}'_s(s) = \frac{1}{\pi}\int\limits_0^\pi \sin\left[s\left(x-\sin x\right)\right]\sin x dx \tag{2.7}$$

is a very simple analytical function which may be approximated quite well by some combination [10] of expansion in powers of s

$$\mathbf{J}'_s(s) = \frac{1+\frac{5}{24}s^2}{2\pi\left(1-s^2\right)}\sin\pi s, \quad s \leq 1 \tag{2.8}$$

and of the asymptotic form

$$\mathbf{J}'_s(s) = \frac{b}{s^{2/3}} - \frac{a}{5s^{4/3}} - \frac{\sin\pi s}{4\pi s^2}, \quad s \gg 1 \tag{2.9}$$

where

$$a = \frac{2^{1/3}}{3^{2/3}\Gamma\left(2/3\right)} \simeq 0.4473, \quad b = \frac{2^{2/3}}{3^{1/3}\Gamma\left(1/3\right)} \simeq 0.41085. \tag{2.10}$$

The map (2.3) is the general mapping form of the classical equations of motion for the one-dimensional hydrogen atom in a microwave field derived in the classical perturbation theory approximation. Some analytical and numerical analysis of this map has been done in Refs. [9–11]. Here we analyze different special cases of the map (2.3).

2.1 High frequency limit

For the relatively high frequencies of the field, $s \gg 1$ ($s \geq 2$), theoretical analysis of the classical dynamics of the one-dimensional hydrogen atom in a microwave field is relatively simple. That is why, the energy changes of the electron, $(E_{j+1} - E_j)$ and $(\varepsilon_{j+1} - \varepsilon_j)$, do not depend on the initial energy ε_j and relative frequency $s \gg 1$. Indeed, using the asymptotic form of the derivative of the Anger function, $\mathbf{J}'_s(s) = b/s^{2/3}$, we have $h\left(\varepsilon_{j+1}\right) = 4b = const.$, $\eta\left(\varepsilon_{j+1}\right) = 0$ and, consequently, the following map

$$\left\{\begin{array}{ll} \varepsilon_{j+1} = & \varepsilon_j - 4\pi b F_0 \varepsilon_0^2 \sin\vartheta_j, \\ \vartheta_{j+1} = & \vartheta_j + 2\pi\varepsilon_{j+1}^{-3/2}. \end{array}\right. \tag{2.11}$$

Note, that scaled classical dynamics according to maps (2.3) and (2.11) depends only on single combination of the field parameters, i.e., on the scaled field strength $F_s = F_0\varepsilon_0^2 = F/\omega^{4/3}$.

By the standard [12, 13] linearization procedure, $\varepsilon_j = \varepsilon_0 + \Delta\varepsilon_j$, in the vicinity of the integer relative frequency (resonance), $s_0 = \varepsilon_0^{-3/2} = m$ with m integer, the map (2.11) may be transformed to the standard (Chirikov) map

$$\left\{\begin{array}{ll} I_{j+1} = & I_j + K\sin\vartheta_j, \\ \vartheta_{j+1} = & \vartheta_j + I_{j+1}. \end{array}\right. \tag{2.12}$$

Here $I_j = -3\pi\Delta\varepsilon_j/\varepsilon_0^{5/2}$ and $K = 12\pi^2 b F_0/\sqrt{\varepsilon_0}$.

From condition of the onset of classical chaos for the standard map, $K \geq K_c \simeq 0.9816$ [12–15], we may, therefore, estimate the threshold field strength for chaotization of dynamics and ionization of the atom in the high frequency field

$$F_0^c = K_c / \left(12\pi^2 b s_0^{1/3}\right) \simeq 0.02 s_0^{-1/3}. \tag{2.13}$$

Some times [6] one writes the map (2.11) for a variable $N = -1/2n^2\omega$, the change of which gives the number of absorbed photons,

$$\begin{cases} N_{j+1} = & N_j + 2\pi \left(F/\omega^{5/3}\right) \sin\vartheta_j, \\ \vartheta_{j+1} = & \vartheta_j + 2\pi\omega \left(-2\omega N_{j+1}\right)^{-3/2}. \end{cases} \tag{2.14}$$

We see that for such variables dynamics of the system depends on two parameters: on the quantum scaled field strength $F_q = F/\omega^{5/3}$ [2, 3] and on the field frequency ω. Map (2.14) is, therefore, not the most convenient one for analysis of the *classical* dynamics.

In general there are, however, no essential difficulties in theoretical analysis of classical nonlinear dynamics of the highly excited hydrogen atom in the microwave field of relative frequency $s_0 = \omega n_0^3 \geq 0.5$ when the field strength is lower or comparable with the threshold field strength for the onset of classical chaos, i.e., if the microwave field is considerably weaker than the characteristic Coulomb field. In such a case, energy change of the electron during the period of intrinsic motion is relatively small and application of the classical perturbation theory for derivation of the Kepler map (2.3) is sufficiently correct. Further analysis of transition to chaotic behavior and of the ionization process may be based on the map (2.3) and for $s_0 \simeq 0.3 \div 1.5$ results in the impressive agreement between measured ionization curves and those obtained from the map (2.3). Even analytical estimation of the threshold field strengths based on this map is rather proper [9–11].

Sufficiently more complicated is analysis of transition to stochastic motion and of ionization process in the region of low relative frequencies, $s_0 \leq 0.3$.

2.2 Low frequency limit

For the low relative frequencies of the microwave field, $s \ll 1$, the map (2.3) may be simplified as well. Using expansion of the function $\mathbf{J}'_s(s)$ in powers of s, $\mathbf{J}'_s(s) \simeq s/2$, for $s \ll 1$ we have according to Eqs. (2.4) and (2.6)

$$\begin{cases} h\left(\varepsilon_{j+1}\right) & = 2/\varepsilon_{j+1}^{5/2} \\ \eta\left(\varepsilon_{j+1}\right) & = -5/\varepsilon_{j+1}^{7/2}. \end{cases} \tag{2.15}$$

Consequently map (2.3) transforms to the form

$$\begin{cases} \varepsilon_{j+1} = & \varepsilon_j - 2\pi F_0 \left(\varepsilon_0^2/\varepsilon_{j+1}^{5/2}\right) \sin\vartheta_j, \\ \vartheta_{j+1} = & \vartheta_j + 2\pi/\varepsilon_{j+1}^{3/2} + 5\pi F_0 \left(\varepsilon_0^2/\varepsilon_{j+1}^{7/2}\right) \cos\vartheta_j. \end{cases} \tag{2.16}$$

This map is a little bit more complicated than map (2.11) for high frequencies, however, it may easily be analyzed as numerically as well as analytically. Note first of all, that energy

change of the electron during the period of intrinsic motion (after one step of iteration), $|\varepsilon_{j+1} - \varepsilon_j|$, is considerably smaller than the binding energy of the electron $\varepsilon_j \simeq \varepsilon_0$ if the field strength is lower or comparable with the threshold field strength, i.e., $2\pi F_0 \left(\varepsilon_0^2/\varepsilon_{j+1}^{5/2}\right) \simeq 2\pi F_0 \varepsilon_0^{-1/2} \ll \varepsilon_0$, or $2\pi F_0 s_0 \ll 1$ if $F_0 \leq F_0^{st} \simeq 0.13$ and $s_0 \ll 1$. This indicates that the map (2.16) is probably suitable for description of dynamics even in the low frequency region where the field is relatively strong.

2.2.1 Adiabatic ionization

For low frequencies, $2\pi s = 2\pi/\varepsilon^{3/2} \ll 1$, according to the second equation of map (2.16) the change of the angle ϑ after one step of iteration is small. As it was noticed above, the energy change is relatively small too. Therefore, we may transform difference equations (2.16) to differential equations of the form

$$\left\{ \begin{array}{ll} \frac{d\varepsilon}{dj} = & -\frac{2\pi\varepsilon_0^2 F_0}{\varepsilon^{5/2}} \sin\vartheta, \\ \frac{d\vartheta}{dj} = & \frac{2\pi}{\varepsilon^{3/2}} + \frac{5\pi\varepsilon_0^2 F_0}{\varepsilon^{7/2}} \cos\vartheta. \end{array} \right. \tag{2.17}$$

Dividing second equation of the system (2.17) by the first one we obtain one differential equation

$$\frac{d\left(\cos\vartheta\right)}{d\varepsilon} = \frac{\varepsilon}{\varepsilon_0^2 F_0} + \frac{5\cos\vartheta}{2\varepsilon}. \tag{2.18}$$

Analytical solution of Eq. (2.18) with the initial condition $\varepsilon = \varepsilon_0$ when $\vartheta = \vartheta_0$ is

$$\cos\vartheta = z^5 \cos\vartheta_0 - 2z^4 \left(1 - z\right)/F_0, \quad z = \sqrt{\varepsilon/\varepsilon_0}. \tag{2.19}$$

Eq. (2.19) describes motion of the system in ε and ϑ variables, i.e., represents functional dependence between two dynamical variables. For relatively low values of F_0, i.e., for $F_0 < \frac{2}{5}z^4 = \frac{2}{5}\left(\frac{\varepsilon}{\varepsilon_0}\right)^2$, the right-hand side of Eq. (2.18) is positive for all phases ϑ. Therefore, $\cos\vartheta$ and ε decrease and increase simultaneously and, according to Eq. (2.19), there is a motion in all interval $[0, 2\pi]$ of the angle ϑ. For $F_0 > \frac{2}{5}z^4$, however, the increase of the angle ϑ in the interval $0 \div \pi$ turns at $\vartheta \simeq \pi$ into the decrease. This results in fast decrease of ε and to ionization process. It is easy to understand from analysis of Eq. (2.19) that the minimal value of F_0 for such a motion (resulting in ionization) corresponds to $\vartheta_0 = 0$ and $\vartheta = \pi$. This value of F_0 is very close to the maximal value of F_0 resulting to the motion in all interval $[0, 2\pi]$ of ϑ, i.e., the maximum of the expression

$$F_0 = 2z\left(1 - z\right) / \left(1 + z^5\right). \tag{2.20}$$

This maximum is at $z = z_0$, where z_0 is a solution of the equation $z^5 + 5z - 4 = 0$, being $z_0 \simeq 0.75193$. The critical value of the relative field strength, therefore, is $F_0^0 = 2z_0^4/5 = 0.1279$ which is only 1 lower the adiabatic ionization threshold $F_0^{st} = 2^{10}/\left(3\pi\right)^4 = 0.1298$.

2.2.2 Chaotic ionization

For higher relative frequencies, $s_0 \geq 0.1$, ionization process is due to chaotic dynamics of the highly excited electron of the hydrogenic atom in a microwave field. There are different

criterions for estimation of the parameters when dynamics of the nonlinear system becomes chaotic. For analysis of transition to chaotic behavior of the motion described by maps (2.3), (2.11) and (2.16) the most proper, to is the criterion related with the chaotization of the phases [13]

$$K = \max \left| \frac{\delta \vartheta_{j+1}}{\delta \vartheta_j} - 1 \right| \geq 1. \tag{2.21}$$

Here max means the maximum with respect to the phase ϑ_j and variation of the phase ϑ_{j+1} with respect to the phase ϑ_j means the full variation including dependence of ϑ_{j+1} on ϑ_j through the variable ε_{j+1} in Eqs. (2.3), (2.11) and (2.16).

Applying criterion (2.21) to the general map (2.3)–(2.4) we obtain the threshold field strength

$$F_0^c = \frac{\varepsilon^{7/2}}{12\pi^2 \varepsilon_0^2 \mathbf{J}'_s(s)}. \tag{2.22}$$

If $\varepsilon \simeq \varepsilon_0$ Eq. (2.22) yields the result

$$F_0^c = \left(12\pi^2 s \mathbf{J}'_s(s)\right)^{-1} \tag{2.23}$$

which for $s \gg 1$ coincides with Eq. (2.13).

For more precise evaluation of the critical field strengths we should take into account the change (increase) of the electron's energy due to the influence of the electromagnetic field [11]. For higher relative frequency s or lower scaled energy ε_j the threshold ionization field is lower. Therefore, if the scaled energy ε_j decreases in a result of relatively regular dynamics in not very strong microwave field, then it is sufficient the lower field strength for transition to the chaotic dynamics.

3 Transition to nonchaotic dynamics and synchronization in randomly driven systems

When an ensemble of bounded in a fixed external potential particles with different initial conditions are driven by an identical sequence of random forces, the ensemble of trajectories may become identical at long times, i.e. synchronization of the identical systems by common noise may be observed. Fahy and Hamann [16] considered a particle of mass m moving according to Newton's equations in a potential $V(x)$, except that at regular time intervals τ the particle is stopped and its velocity is reset to random value chosen from a Maxwell distribution with temperature T . It should be stressed that for every particle of the ensemble it was given an identical, randomly chosen velocity at the start of each step of time length τ. This motion is in many respects similar to Brownian motion of the particles at a temperature T. However, if the time interval τ between stops is lower than a threshold value τ_c, the final trajectories of the particles are independent on the initial conditions; all trajectories become point by point identical in time. Although the trajectory is highly erratic and random, the system is not chaotic.

The similar effect may also be observed in a more general and realistic (from the physical point of view) case, i.e., when mixing at time intervals τ some part α of the old velocity

$\mathbf{v}^{old}$ with random velocity $\mathbf{v}^{ran}$ to get a new starting velocity $\mathbf{v}^{new} = \alpha\mathbf{v}^{old} + \mathbf{v}^{ran}$. Here a threshold value τ_c depends on α.

Let us consider a particle of mass m moving in a one-dimensional potential $V(x)$ which confines particles to a finite region. At a time intervals τ_i the particle is partially stopped and its velocity is reset to a new starting velocity $v_i = \alpha v_i^{old} + v_i^{ran}$. Between the stops the particle moves according to Newton's equations

$$\frac{dx}{dt} = v, \qquad \frac{d^2x}{dt^2} = -\frac{1}{m}\frac{dV}{dx}. \tag{3.1}$$

When two particles initially at points x_0 and x_0' are started with velocities v_0 and v_0' and are driven by an identical sequence of random velocities v_i^{ran} at the same time intervals τ_i, coordinates and velocities of them may accidentally draw closer to one another. The convergence of the two trajectories to the single final trajectory will depend on the evolution with a time of the small variances of the distance $\Delta x_i = x_i' - x_i$ and velocity $\Delta v_i = v_i' - v_i$. Moreover, we investigate a transition from chaotic to nonchaotic behavior. Generally, such a transition may be detected from analysis of behavior of the neighboring trajectories and it is described by the Lyapunov characteristic exponents and KS metric entropy of the flow of trajectories in a given region of phase space [12, 13, 17–20].

From formal solutions $x = x(x_i, v_i, t)$ and $v = v(x_i, v_i, t)$ of equations (3.1) with initial conditions $x = x_i$ and $v = v_i$ at $t = 0$ it follows an equation for $\Delta x(t)$ and $\Delta v(t)$ at a time moment t:

$$\begin{pmatrix} \Delta x(t) \\ \Delta v(t) \end{pmatrix} = \mathbf{T}(\alpha; x_i, v_i, t) \begin{pmatrix} \Delta x_i \\ \Delta v_i \end{pmatrix} \tag{3.2}$$

where the matrix $\mathbf{T}$ is of the form

$$\mathbf{T} = \begin{pmatrix} T_{xx} & \alpha T_{xv} \\ T_{vx} & \alpha T_{vv} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x(x_i, v_i, t)}{\partial x_i} & \alpha \dfrac{\partial x(x_i, v_i, t)}{\partial v_i} \\ \dfrac{\partial v(x_i, v_i, t)}{\partial x_i} & \alpha \dfrac{\partial v(x_i, v_i, t)}{\partial v_i} \end{pmatrix}. \tag{3.3}$$

Note, that the similar method of investigation is used in the theory of transition to chaos in classical systems [17, 18]. However, the motion in the form (3.2) and (3.3) is represented as the non-area-preserving tangent map, while classical dynamics of the conservative systems may be represented by the area-preserving maps.

According to equations (3.1) and (3.3) matrix elements T_{xx} and T_{xv} satisfy the equation

$$\frac{d^2T_x}{dt^2} = -\frac{1}{m}\frac{d^2V}{dx^2}\bigg|_{x=x(x_i,v_i,t)} T_x \tag{3.4}$$

while $T_{vx} = \dot{T}_{xx},\ T_{vv} = \dot{T}_{xv}$ and the initial conditions at $t = 0$ are:

$$T_{xx}(x_i, v_i, 0) = T_{vv} = 1, \quad T_{xv} = T_{vx} = 0,$$

$$\dot{T}_{xx}(x_i, v_i, 0) = \dot{T}_{vv} = 0,\ \dot{T}_{xv} = 1,\ \dot{T}_{vx} = -\frac{1}{m}\frac{d^2V}{dx^2}\bigg|_{x=x_i}. \tag{3.5}$$

Therefore, the dynamics of the distance between the particles Δx and the difference of the velocity Δv may be represented by the non-area-preserving mapping form of the equations of motion

$$\begin{pmatrix} \Delta x_{i+1} \\ \Delta v_{i+1} \end{pmatrix} = \mathbf{T}(\alpha; x_i, v_i, \tau_i) \begin{pmatrix} \Delta x_i \\ \Delta v_i \end{pmatrix}. \tag{3.6}$$

In general, the intervals between the resets of the velocity τ_i may be depending on the number of step i.

Further analysis of the model may be based on the general theory of the dynamics of classical systems represented as maps [17–19]. Thus, for $\alpha = 0$ the Lyapunov exponent is defined as

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\tau_i} \ln\Big|T_{xx}(x_i, v_i, \tau_i)\Big| \tag{3.7}$$

and may be easily evaluated numerically.

For $\alpha = 1$ the map (3.6) is area-preserving and $\det \mathbf{T}(1; x_i, v_i, \tau_i) = 1$, while in general $\det \mathbf{T} = \alpha$, $Tr\mathbf{T} = T_{xx} + \alpha T_{vv}$ and the eigenvalues $\mu_{1,2}$ of the $\mathbf{T}$ matrix are given by equation

$$\mu^2 - \mu Tr\mathbf{T} + \det \mathbf{T} = 0$$

which yields

$$\mu_{1,2} = \frac{1}{2}\Big[T_{xx} + \alpha T_{vv} \mp \sqrt{(T_{xx} + \alpha T_{vv})^2 - 4\alpha}\Big].$$

So, the eigenvalues come in reciprocal pair, $\mu_1\mu_2 = \alpha$. For $(T_{xx} + \alpha T_{vv})^2 - 4\alpha < 0$ the eigenvalues form a complex conjugate pair with $|\mu_1| = |\mu_2| = \sqrt{\alpha}$, otherwise the eigenvalues are real.

Generally, the mapping $\mathbf{T}(\alpha; x_i, v_i, \tau_i)$ in (3.6) is depending on the starting coordinates x_i and v_i. Therefore, calculation of the mapping for n steps, $\mathbf{T}_n = \mathbf{T}(\alpha; x_{i+n-1}, v_{i+n-1}, \tau_{i+n-1}) \cdot \mathbf{T}(\alpha; x_{i+n-2}, v_{i+n-2}, \tau_{i+n-2}) \ \cdots \ \mathbf{T}(\alpha; x_i, v_i, \tau_i)$, and of the corresponding eigenvalues are complicated problems. Further we will evaluate the averaged quantities

$$\sigma_{1,2} = \langle \frac{1}{\tau_i} \ln|\mu_{1,2}| \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\tau_i} \ln\Big|\mu_{1,2}(x_i, v_i, \tau_i)\Big| \tag{3.8}$$

which are analogous of the averaged Lyapunov exponent (3.7), characterize the rate of the exponential increase of the separation of the two initially adjacent points and are related with the KS entropy of the system [17, 18]. Comparisons of the threshold values τ_c from the direct numerical simulations with those from the criterion

$$\sigma_{largest} = 0 \tag{3.9}$$

indicate to the usefulness of the quantities (3.8) for analysis of transition from nonchaotic to chaotic behavior and synchronization of the systems [21–23].

Therefore, theoretical analysis based on the mapping form of equations of motion for the distance between the particles and the difference of the velocity allows to simplify the problem of investigation of transition to nonchaotic behavior and results in the expressions for the criteria of the nonchaotic motion. Theoretical results agree well with the direct numerical simulations and indicate to the possibilities of generalization of the model, e.g. to

more degrees of freedom, for random values of the time intervals between the resets of the velocity and for systems driven by the random forces [21–23]. In paper [24] this method has been generalized and used for analysis of systems with repulsive force between particles, some scaling properties for the threshold reset time have been derived and it has been suggested that such convergence of chaotic orbits is a rather general phenomenon.

4 Dynamics of quantum systems

4.1 Two-level system

Let's consider the simplest quantum dynamical process and the influence of frequent measurements on the outcome of the dynamics. Time evolution of the amplitudes $a_1(t)$ and $a_2(t)$ of the two-state wave function

$$\Psi = a_1(t)\Psi_1 + a_2(t)\Psi_2 \tag{4.1}$$

of the system in the resonance field (in the rotating wave approximation) or of the spin-half system in a constant magnetic field may be represented as

$$\begin{cases} a_1(t) = & a_1(0)\cos\frac{1}{2}\Omega t + ia_2(0)\sin\frac{1}{2}\Omega t \\ a_2(t) = & ia_1(0)\sin\frac{1}{2}\Omega t + a_2(0)\cos\frac{1}{2}\Omega t, \end{cases} \tag{4.2}$$

where Ω is the Rabi frequency. We introduce a matrix $\mathbf{A}$ representing time evolution during the time interval τ (between time moments $t = k\tau$ and $t = (k+1)\tau$ with integer k) and rewrite Eq. (4.2) in the mapping form

$$\begin{pmatrix} a_1(k+1) \\ a_2(k+1) \end{pmatrix} = \mathbf{A} \begin{pmatrix} a_1(k) \\ a_2(k) \end{pmatrix} \tag{4.3}$$

where the evolution matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} \cos\varphi & i\sin\varphi \\ i\sin\varphi & \cos\varphi \end{pmatrix}, \quad \varphi = \frac{1}{2}\Omega\tau. \tag{4.4}$$

Evidently, the evolution of the amplitudes from time $t = 0$ to $t = T = n\tau$ may be expressed as

$$\begin{pmatrix} a_1(n) \\ a_2(n) \end{pmatrix} = \mathbf{A}^n \begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix}. \tag{4.5}$$

One may calculate matrix $\mathbf{A}^n$ by the method of diagonalization of the matrix $\mathbf{A}$. The result naturally is

$$\mathbf{A}^n = \begin{pmatrix} \cos n\varphi & i\sin n\varphi \\ i\sin n\varphi & \cos n\varphi \end{pmatrix}. \tag{4.6}$$

Note that $n\varphi = \frac{1}{2}\Omega T$.

Equations (4.2)–(4.6) represent time evolution of the system without the intermediate measurements in the time interval $0 \div T$. If at $t = 0$ the system was in the state Ψ_1, i.e. $a_1(0) = 1$ and $a_2(0) = 0$, and if $\Omega T = \pi$ then at the time moment $t = T$ we would certainly

find the system in the state Ψ_2, i.e. it would be $|a_1(T)|^2 = 0$ and $|a_2(T)|^2 = 1$, a certain transition between the states.

Let's consider now the dynamics of the system with the intermediate measurements every time interval τ. Measurement of the system's state in the time moment $t = k\tau$ projects the system into the state Ψ_1 with the probability $p_1(k) =\mid a_1(k) \mid^2$ or into the state Ψ_2 with the probability $p_2(k) =\mid a_2(k) \mid^2$. After the measurement we know the probabilities $p_1(k)$ and $p_2(k)$ but we have no information about the phases $\alpha_1(k)$ and $\alpha_2(k)$ of the amplitudes

$$a_1(k) = |a_1(k)|\, e^{i\alpha_1(k)}, \quad a_2(k) = |a_2(k)|\, e^{i\alpha_2(k)}, \tag{4.7}$$

i.e. the phases $\alpha_1(k)$ and $\alpha_2(k)$ after every act of the measurement are random [25]. This results in the equation for the probabilities

$$\begin{pmatrix} p_1(k+1) \\ p_2(k+1) \end{pmatrix} = \mathbf{M} \begin{pmatrix} p_1(k) \\ p_2(k) \end{pmatrix}, \tag{4.8}$$

where

$$\mathbf{M} = \begin{pmatrix} \cos^2\varphi & \sin^2\varphi \\ \sin^2\varphi & \cos^2\varphi \end{pmatrix} \tag{4.9}$$

is the evolution matrix for the probabilities. The full evolution from the initial time $t = 0$ until $t = T$ with the $(n-1)$ equidistant intermediate measurement is described by the equation

$$\begin{pmatrix} p_1(n) \\ p_2(n) \end{pmatrix} = \mathbf{M}^n \begin{pmatrix} p_1(0) \\ p_2(0) \end{pmatrix}. \tag{4.10}$$

The result of calculation of the matrix $\mathbf{M}^n$ by the method of diagonalization of the matrix $\mathbf{M}$ is

$$\mathbf{M}^n = \frac{1}{2} \begin{pmatrix} 1+\cos^n 2\varphi & 1-\cos^n 2\varphi \\ 1-\cos^n 2\varphi & 1+\cos^n 2\varphi \end{pmatrix}. \tag{4.11}$$

From Eqs. (4.10) and (4.11) we recover the quantum Zeno effect [25-27]: if initially the system is in the state Ψ_1, than the result of the evolution until the time moment $T = n\tau = \pi/\Omega$ (after the π -pulse) with the $(n-1)$ intermediate measurement will be characterized by the probabilities $p_1(T)$ and $p_2(T)$ for finding the system in the states Ψ_1 and Ψ_2 respectively:

$$\begin{cases} p_1(T) = \frac{1}{2}(1+\cos^n 2\varphi) \simeq \frac{1}{2}(1+e^{-\frac{\pi^2}{2n}}) \simeq 1 - \frac{\pi^2}{4n} \to 1, \\ p_2(T) = \frac{1}{2}(1-\cos^n 2\varphi) \simeq \frac{1}{2}(1-e^{-\frac{\pi^2}{2n}}) \simeq \frac{\pi^2}{4n} \to 0, \ n \to \infty. \end{cases} \tag{4.12}$$

We see that results of equations (4.10)–(4.12) represent the inhibition of the quantum dynamics by measurements and coincide with those obtained by the density matrix technique [26, 27]. This also confirms correctness of the proposition that the act of the measurement may be represented as randomization of the amplitudes' phases. Further we will use this proposition and the same method for the analysis of the repeated measurement influence for the quantum dynamics of multilevel systems which classical counterparts exhibit chaos. We restrict ourselves to the strongly driven by a periodic force systems with one degree of freedom. The investigation is also based on the mapping equations of motion for such systems.

4.1.1 Dynamics of multilevel systems

In general the classical equations of motion are nonintegrable and the Schrödinger equation for strongly driven systems may not be solved analytically. However, mapping forms of the classical and quantum equations of motion greatly facilitates the investigation of stochasticity and quantum–classical correspondence for the chaotic dynamics. From the standpoint of an understanding of the manifestation of the measurements for the dynamics of the multilevel systems the region of large quantum numbers is of greatest interest. Here we may use the quasiclassical approximation and convenient variables are the angle θ and the action I. One of the simplest systems in which the dynamical chaos and its quantum localization may be observed is a system with one degree of freedom described by the unperturbed Hamiltonian $H_0(I)$ and driven by periodic kicks. The full Hamiltonian H of the driven system may be represented as

$$H(I,\theta,t) = H_0(I) + k\cos\theta\sum_j \delta\left(t - j\tau\right) \tag{4.13}$$

where τ and k are the period and strength of the perturbation, respectively.

For the derivation of the quantum equations of motion we expand the state function $\psi(\theta,t)$ of the system through the quasiclassical eigenfunctions, $\varphi_n(\theta) = e^{in\theta}/\sqrt{(2\pi)}$, of the Hamiltonian H_0,

$$\psi(\theta,t) = (2\pi)^{-1/2}\sum_n a_n(t) i^{-n} e^{-in\theta}. \tag{4.14}$$

Here the phase factor i^{-n} is introduced for the maximal simplification of the quantum map. Integrating the Schrödinger equation over the period τ, we obtain the following maps for the amplitudes before the appropriate kicks

$$a_m(t_{j+1}) = e^{-iH_0(m)\tau}\sum_n a_n(t_j) J_{m-n}(k), \quad t_j = j\tau \tag{4.15}$$

where $J_m(k)$ is the Bessel function.

The form (4.15) of the map for the quantum dynamics is rather common: similar maps may be derived for the monochromatic perturbations as well, e.g. for an atom in a microwave field [28]. A particular case of map (4.15), corresponding to the model of a quantum rotator $H = I^2/2$, has been comprehensively investigated with the aim of determining the relationship between classical and quantum chaos. It has been shown that under the onset of dynamical chaos at $K \equiv \tau k > K_c = 0.9816$, motion with respect to I is not bounded and it is of a diffusion nature in the classical case, while in the quantum description diffusion with respect to m is bounded, i.e. the diffusion ceases after some time and the state of the system localizes exponentially. This phenomenon turns out to be typical for models (4.15) with nonlinear Hamiltonians $H_0(I)$ and for other quantum systems. The quantum interference effect is essential for such dynamics and it results in the quantum evolution being quantitatively different from the classical motion. Quantum equations of motion, i.e. the Schrödinger equation and the maps for the amplitudes, are linear equations with respect to the wave function and probability amplitudes. Therefore, the quantum interference effect manifests itself even for quantum dynamics of the systems, the classical counterparts of which are described by nonlinear equations; chaotic dynamics of the later exhibit a dynamical chaos. On the other hand, quantum equations of motion are very complex as well. Thus, the Schrödinger equation is a partial differential equation with the coordinate and time dependent coefficients,

while the system of equations for the amplitudes is the infinite system of equations. Moreover, for the nonlinear Hamiltonian $H_0(m)$ the phases' increments, $H_0(m)\tau$, during the free motion between two kicks while reduced to the basic interval $[0, 2\pi]$ are the pseudorandom quantities as a function of the state's quantum number m. This causes a very complicated and irregular quantum dynamics of the classically chaotic systems. We observe not only very large and apparently irregular fluctuations of probabilities of the states' occupation but also almost irregular fluctuations in time of the momentum dispersion [25, 28].

However, the quantum dynamics of such driven by the external periodic force systems is coherent and the evolution of the amplitudes $a_m(t_{j+1})$ in time is linear: they are defined by the linear map (4.15) with the time independent coefficients. The nonlinearity of the Hamiltonian $H_0(I)$, being the reason of the classical chaos, causes the pseudorandom nature of the increments of the phases, $H_0(m)\tau$, as a function of the state's number m (but constant in time). These increments of the phases remain the same for each kick. So, the dynamics of the amplitudes $a_m(t_{j+1}) = |a_m(t_{j+1})|\, e^{i\alpha_m(t_{j+1})}$ and of their phases, $\alpha_m\,(t_{j+1})$, is strongly deterministic and non-chaotic but very complicated and apparently irregular. For instance, the phases $\alpha_m\,(t_{j+1})$ are phases of the complex amplitudes, $a_m(t_{j+1})$, which are linear combinations (4.15) of the complex amplitudes, $a_n(t_j)$, before the preceding kick with the pseudorandom coefficients, $e^{-iH_0(m)\tau}J_{m-n}(k)$. Nevertheless, the iterative equation (4.15) is a *linear transformation with coefficients regular in time.* That is why, we observe for such dynamics the quasiperiodic reversible in the time evolution [28] with the quantum localization of the pseudochaotic motion.

In paper [28] it has been demonstrated that this peculiarity of the pseudochaotic quantum dynamics is indeed due to the pseudorandom nature of the phases, $H_0(m)\tau$, in Eq. (4.15) as a function of the eigenstate's quantum number m (but not of the kick's number j). Replacing the multipliers $\exp\left[-iH_0(m)\tau\right]$ in Eq. (4.15) by the expressions $\exp\left[-i2\pi g_m\right]$, where g_m is a sequence of random numbers that are uniformly distributed in the interval $[0, 1]$, we observe the quantum localization as well [28]. The essential point here is the independence of the phases $H_0(m)\tau$ or $2\pi g_m$ on the step of iteration j or time t.

4.1.2 Influence of repetitive measurement on the quantum dynamics

Each measurement of the system's state projects the state into one of the energy state φ_m with the definite m. Therefore, if we make a measurement of the system after the kick j but before the next $(j + 1)$ kick we will find the system in the states φ_m with the appropriate probabilities $p_m(j) = |a_m(t_j)|^2$.

In the calculations of the system's dynamics the influence of the measurements may be taken into account through randomization of phases of the amplitudes after the measurement of the appropriate state's populations. The phases of amplitudes after the measurement are completely random and uncorrelated with the phases before the measurement, after another measurements and with the phases of other measured or unmeasured states. Therefore, after the full measurement of the system after the kick j, all phases of the amplitudes $a_m(t_j)$ are random. So, this full measurement of the system's state influences on the further dynamics of the system through the randomization of the phases of amplitudes. This fact may be expressed by replacement in Eqs. (4.15) of the amplitudes $a_m(t_{j+1})$ by the amplitudes $e^{i\beta_m(t_{j+1})}a_m(t_{j+1})$ with the random phases $\beta_m(t_{j+1})$. The essential point here is that the

phases $\beta_m(t_{j+1})$ are different, uncorrelated for the different measurements, i.e. for different time moments of the measurement t_{j+1}. This is the principal difference of the random phases $\beta_m(t_{j+1})$ from the phases $H_0(m)\tau$ in Eqs. (4.15) which are pseudorandom variables as functions of the eigenstate's quantum number m (but not of the time moment t_{j+1}).

Instead of representing the detailed quantum dynamics expressed as the evolution of all amplitudes in the expansion of the wave function (4.14) we may represent only dynamics of the momentum dispersion $\langle (m_j - m_0)^2 \rangle = \sum\limits_m (m - m_0)^2 \left| a_m (t_j) \right|^2$ where m_0 is the initial momentum (quantum number). Such a representation of the dynamics is simpler, more picturesque and more comfortable for comparison with the classical dynamics.

Theoretically differences of dynamics without measurement and with the measurement may be understood from the iterative equations for the momentum dispersion:

$$\left\langle (m_{j+1} - m_0)^2 \right\rangle = \sum_m (m - m_0)^2 \left| a_m (t_{j+1}) \right|^2 , \tag{4.16}$$

where

$$\left| a_m (t_{j+1}) \right|^2 = \sum_{n,n'} J_{m-n}(k) J_{m-n'}(k) a_n (t_j) a^*_{n'} (t_j) . \tag{4.17}$$

Substitution of Eq. (4.17) into Eq. (4.16) yields

$$\begin{array}{c} \langle (m_{j+1} - m_0)^2 \rangle = \sum\limits_{m,n} (m - m_0)^2 J^2_{m-n}(k) \left| a_n (t_j) \right|^2 \\ +2 \sum\limits_{m,n} \sum\limits_{n'<n} (m - m_0)^2 J_{m-n}(k) J_{m-n'}(k) Re \left[a_n (t_j) a^*_{n'} (t_j) \right] . \end{array} \tag{4.18}$$

For the random phase differences of the amplitudes $a_n (t_j)$ and $a^*_{n'} (t_j)$ with $n' \neq n$, which is a case after the measurement of the system's state, the second term of Eq. (4.18) on the average equals zero. Then from Eq. (4.18) we have

$$\begin{array}{l} \langle (m_{j+1} - m_0)^2 \rangle = \sum\limits_n \left| a_n (t_j) \right|^2 \sum\limits_m (m - m_0)^2 J^2_{m-n}(k) \\ = \sum\limits_m \left| a_m (t_j) \right|^2 \left(m^2 - m_0^2 + \frac{k^2}{2} \right) = \langle (m_j - m_0)^2 \rangle + \frac{k^2}{2} . \end{array} \tag{4.19}$$

In the derivation of Eq. (4.19) we have used the summation expressions

$$\sum_m m J^2_{m-n}(k) = 0 \quad \text{and} \quad \sum_m m^2 J^2_{m-n}(k) = n^2 + \frac{k^2}{2} . \tag{4.20}$$

Therefore, according to Eq. (4.19) for the uncorrelated phases of the amplitudes $a_n (t_j)$ and $a^*_{n'} (t_j)$ with $n' \neq n$ the dispersion of the momentum as a result of every kick increases on the average in the magnitude $k^2/2$, the same as for the classical dynamics. Thus, we reveal that repetitive measurement of the multilevel systems with quantum suppression of classical chaos results in the delocalization of the states superposition and restoration of the chaotic dynamics. Since this effect is reverse to the quantum Zeno effect we have called this phenomenon the *'quantum anti-Zeno effect'* [25].

5 Conclusions

In this paper on the concrete examples we have demonstrated usefulness of the mapping equations for analysis of dynamics the nonlinear systems: transition to chaotic behavior in Hamiltonian systems, synchronization of chaotic systems driven by identical noise and effect of repetitive measurements for quantum dynamics.

References

1. L. D. Landau and E. M. Lifshitz, *Classical Field Theory* (Pergamon, New York, 1975).

2. B. Kaulakys, V. Gontis, G. Hermann, and A. Scharmann, Phys. Lett. A **159**, 261 (1991); B. Kaulakys, Acta Phys. Pol. B **23**, 313 (1992).

3. B. Kaulakys, V. Gontis, and G. Vilutis, Lith. Phys. J. (Allerton Press, Inc.) **33**, 290 (1993); B. Kaulakys and G. Vilutis, in *AIP Conf. Proc.* (AIP, New York) **329**, 389 (1995); e-print archives: xxx.lanl.gov/quant-ph/9504007.

4. N. B. Delone, V. P. Krainov, and D. L. Shepelyansky, Usp. Fiz. Nauk **140**, 355 (1983) [Sov. Phys.-Usp. **26**, 551 (1983)].

5. G. Casati, B. V. Chirikov, D. L. Shepelyansky, and I. Guarneri, Phys. Rep. **154**, 77 (1987).

6. G. Casati, I. Guarneri, and D. L. Shepelyansky, IEEE J. Quantum Electron. **24**, 1420 (1988).

7. R. V. Jensen, S. M. Susskin, and M. M. Sanders, Phys. Rep. **201**, 1 (1991).

8. P. M. Koch, in *Chaos and Quantum Chaos*, edited by W. Heiss, Lecture Notes in Physics Vol. **411** (Springer-Verlag, Berlin, 1992), p. 167; P. M. Koch and K. A. H. van Leeuwen, Phys. Rep. **255**, 289 (1995).

9. V. Gontis and B. Kaulakys, Deposited in VINITI as No.5087-V86 (1986) and Lit. Fiz. Sb. **27**, 368 (1987) [Sov Phys.-Collect. **27**, 111 (1987)].

10. V. Gontis and B. Kaulakys, J. Phys. B: At. Mol. Opt. Phys. **20**, 5051 (1987).

11. B. Kaulakys and G. Vilutis, in *Chaos - The Interplay between Stochastic and Deterministic Behaviour,* edited by P. Garbaczewski, M. Wolf, and A. Weron, Lecture Notes in Physics Vol. **457** (Springer-Verlag, Berlin, 1995), p. 445; chao-dyn/9503011; B. Kaulakys, D. Grauzhinis and G. Vilutis, Europhys. Lett. **43**, 123 (1998); physics/9808048.

12. A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983 and 1992).

13. G. M. Zaslavskii, *Stochastic Behavior of Dynamical Systems* (Nauka, Moscow, 1984; Harwood, New York, 1985).

14. R. V. Jensen, Am. Scient. **75**, 168 (1987).

15. B. V. Chirikov, Phys. Reports **52**, 265 (1979).

16. S. Fahy and D. R. Hamann, Phys. Rev. Lett. **69**, 761 (1992).

17. D. Park, *Classical Dynamics and Its Quantum Analogues* (Springer, Berlin, 1990).

18. L. E. Reichl, *The Transition to Chaos: In Conservative Classical Systems: Quantum Manifestations* (Springer, Berlin, 1992).

19. J. D. Meiss, Rev. Mod. Phys. **64**, 795 (1992).

20. J. M. Ottino *et al.*, Science **257**, 754 (1992).

21. B. Kaulakys and G. Vektaris, Phys. Rev. A **52**, 2091 (1995); chao-dyn/9504009.

22. F. Ivanauskas, T. Meškauskas and B. Kaulakys, *New Trends in Prob. and Stat.*, Vol. 4, Eds. A. Laurinčikas *et al.* (TEV, Vilnius, 1997), p. 467.

23. B. Kaulakys, F. Ivanauskas and T. Maškauskas, *Proc. Intern. Conf. on Nonlinearity, Bifurcation and Chaos: the Doors to the Future*, Lodz-Dobiesckow, Sept. 16-18, 1996; chao-dyn/9610020; Intern. J. Bifurcation and Chaos (to be published).

24. Y.-Y. Chen, Phys. Rev. Lett. **77**, 4318 (1996).

25. B. Kaulakys and V. Gontis, Phys. Rev. A **56**, 1131 (1997); quant-ph/9708024.

26. R. J. Cook, Phys. Scr. **T21**, 49 (1988).

27. W. M. Itano, D. J. Heinzen, J. J. Bollinger, and D. J. Wineland, Phys. Rev. A **41**, 2295 (1990).

28. V. G. Gontis and B. P. Kaulakys, Liet. Fiz. Rink. **28**, 671 (1988) [Sov. Phys.-Collec. **28**(6), 1 (1988)].